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Energy density and packet formation in a vibrating cavity

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Abstract. We give analytical expressions for the energy density of the massless scalar field excited from an arbitrary initial state (including vacuum) due to the non-stationary Casimir effect in an ideal one-dimensional cavity with vibrating walls, provided the frequency of vibrations is close to a multiple frequency of the fundamental unperturbed field mode. The formation of sharp packets is studied in detail. The influence of the initial state (vacuum, thermal and coherent) on the form of the packets is demonstrated.

1. Introduction

Classical and quantum phenomena in cavities with moving boundaries have attracted the attention of many researchers for a long time. The first solutions of the classical wave equation in the one-dimensional (1D) space domain confined with boundaries moving with *constant* velocities were obtained as far back as in [1, 2]; later they were rediscovered and generalized in [3–5] (a detailed reference list can be found in [6]). Moore's paper [7] seems to be the first one devoted to the quantum aspects of the problem. This topic became particularly popular in the last decade, being known now under the following names: the *non-stationary Casimir effect* [8], the *dynamical Casimir effect* [9] or *mirror- (motion) induced radiation* [10, 11]). For the most recent achievements in this area and references to other works see, e.g., [12–26]. One of the important theoretical results obtained over the last few years was the prediction of the exponential growth of the energy of the field under the resonance conditions, when the wall performs vibrations at a frequency which is a multiple of the unperturbed field eigenfrequency [6, 27–32] (similar results for the classical fields and strings were obtained in [33–35]).

We address the problem of the *energy density distribution* inside a cavity with oscillating boundaries. It was indicated in [28, 30, 31, 36, 37] (and earlier in [38] for the classical field) that the main part of the energy is concentrated in several sharp peaks which move from one boundary to another, becoming narrower and narrower over the course of time. However, until now the evolution of these packets was studied within the framework of numerical calculations only, and no explicit analytical expressions for the shape of the packets were found.

Recently, the analytical solutions to the problem of a one-dimensional ideal cavity with *resonantly vibrating* boundaries were found [18]. These solutions enabled one to account for the effects of detuning from a strict resonance (showing that no photons can be created unless the dimensionless detuning is smaller than the dimensionless amplitude of the boundary oscillations [18]), to study the effect of squeezing and to calculate the photon distribution functions in all the cavity modes [24]. Here we use the solutions given in [18] to provide

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simple explicit formulae for the shape, height and width of the energy packets formed in the vibrating cavity, for any initial state of the field, including as examples the vacuum, thermal and coherent states.

2. Field operator in a 1D cavity with oscillating boundaries

We consider the model of a massless scalar field in a 1D cavity formed by two infinite ideal plates whose positions are given by $x_{left} \equiv 0$ and $x_{right} = L(t)$, where $L(t)$ is a given function (for generalizations to a generic case of two moving boundaries see [14, 18, 37]). The field operator $\hat{A}(x,t)$ *in the Heisenberg representation* can be written in the form (hereafter $c = \hbar = 1$)

$$\hat{A}(x,t) = \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} \left[\hat{b}_n \psi^{(n)}(x,t) + \text{h.c.}\right] \qquad \left[\hat{b}_n, \hat{b}_k^{\dagger}\right] = \delta_{nk}. \tag{2.1}$$

This operator must satisfy the wave equation

$$\hat{A}_{tt} - \hat{A}_{xx} = 0 \tag{2.2}$$

and the boundary conditions [7]

$$\hat{A}(0,t) = \hat{A}(L(t),t) = 0. \tag{2.3}$$

At $t < 0$, when the wall is assumed to be at rest, the mode function $\psi^{(n)}(x,t)$ in (2.1) has a simple factorized form

$$\psi_0^{(n)}(x,t) = \mathrm{e}^{-\mathrm{i}\omega_n t} \sin(\pi n x/L_0) \qquad \omega_n = n\pi/L_0. \tag{2.4}$$

The normalization factor $2/\sqrt{n}$ in (2.1) is chosen in such a way that the energy of the field in the stationary case can be represented as a sum of energies of independent mode oscillators (see the next section).

There exist two approaches to the problem. The first one is usually related to the paper by Moore [7], although it was used long before (in the context of the classical problems): see, e.g., [1–4] and the reference list in [6]. In this approach, the mode function $\psi^{(n)}(x,t)$ is taken in such a form that the *wave equation* (2.2) is satisfied automatically:

$$A_n(x,t) = \{\exp[-\mathrm{i}\pi n R(t-x)] - \exp[-\mathrm{i}\pi n R(t+x)]\}/(2\mathrm{i}). \tag{2.5}$$

Then the boundary conditions (2.3) result in the *functional equation* for the function $R(\xi)$

$$R(t + L(t)) - R(t - L(t)) = 2. \tag{2.6}$$

For the *oscillating boundary*, the approximate solutions to equation (2.6) in the form of a rather straightforward expansion over the small amplitude of oscillations were obtained in [7, 39, 40]. However, these simple solutions cannot be used in the resonance case, due to the presence of secular terms. Different non-trivial approximate solutions in this case were found in [38, 41, 42]. They were generalized and specified in [28, 30, 36, 37]. However, the structure of these solutions is rather complicated, so it was difficult to use them to obtain explicit analytical expressions for the shape of the packets emerging in the cavity.

Another method [6, 27] is based on the expansion of each function $\psi^{(n)}(x,t)$ in a series with respect to the *instantaneous basis*

$$\psi^{(n)}(x,t) = \sqrt{\frac{L(0)}{L(t)}} \sum_{k=1}^{\infty} Q_k^{(n)}(t) \sin\left[\frac{\pi k x}{L(t)}\right] \tag{2.7}$$

so that the *boundary conditions* (2.3) are satisfied automatically. Then the wave equation (2.2) can be replaced by an infinite set of coupled ordinary differential equations for the coefficients $Q_k^{(n)}(t)$

$$\ddot{Q}_k^{(n)} + \omega_k^2(t)\,Q_k^{(n)} = 2\sum_{j=1}^{\infty} g_{kj}(t)\,\dot{Q}_j^{(n)} + \sum_{j=1}^{\infty} \dot{g}_{kj}(t)\,Q_j^{(n)} + \mathcal{O}\left(g_{kj}^2\right) \tag{2.8}$$

where

$$\omega_k(t) = k\pi/L(t) \qquad g_{kj} = -g_{jk} = (-1)^{k-j}\frac{2kj\dot{L}(t)}{\left(j^2-k^2\right)L(t)}.$$

In the case of the oscillating boundary

$$L(t) = L_0\,(1+\varepsilon\sin\left[p\omega_1(1+\delta)t\right]) \qquad \omega_1 = \pi c/L_0 \quad p = 1, 2, \ldots \tag{2.9}$$

the set of equations (2.8) can be simplified under the resonance condition $|\delta| \ll 1$ and for the small amplitude of oscillations $|\varepsilon| \ll 1$, if one writes

$$Q_k^{(n)}(t) = \rho_k^{(n)} \mathrm{e}^{-\mathrm{i}\omega_k(1+\delta)t} - \rho_{-k}^{(n)} \mathrm{e}^{\mathrm{i}\omega_k(1+\delta)t} \tag{2.10}$$

assuming the coefficients $\rho_k^{(n)}$ ($k = \pm1, \pm2, \ldots$; $n = 1, 2, \ldots$) to be slowly varying functions of time, whose derivatives are proportional to the small parameter ε (the frequency ω_k was defined in equation (2.4)). Putting (2.10) into equation (2.8) and neglecting the second-order terms like $\ddot{\rho}_k^{(n)} \sim \varepsilon^2$, one obtains, after averaging over fast oscillations with the multiple frequencies of ω_1, the equations [18]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_k^{(n)} = \sigma\left[(k+p)\rho_{k+p}^{(n)} - (k-p)\rho_{k-p}^{(n)}\right] + 2\mathrm{i}\gamma k\rho_k^{(n)} \tag{2.11}$$

where

$$\gamma \equiv \delta/\varepsilon \qquad \sigma \equiv (-1)^p \qquad \tau \equiv \tfrac{1}{2}\varepsilon\omega_1 t. \tag{2.12}$$

Looking for the functions $Q_k^{(n)}(t)$ in the form (2.10) we neglect the terms proportional to the higher harmonics $\exp\left[\pm\mathrm{i}r\omega_k t\right]$, $r = 2, 3, \ldots$. However, the amplitudes of these corrections are very small, since they are of the order of ε (or less). The approximation (2.10) cannot be justified for the values of time exceeding $t_2 \sim \left(\omega_1\varepsilon^2\right)^{-1}$, when the neglected terms of the order of ε^2 could become essential. However, for the realistic values $\omega_1 \sim 10^{10}\ \mathrm{s}^{-1}$ and $\varepsilon \sim 10^{-8}$ corresponding to the possible experimental realizations [6] the limiting time t_2 is of the order of weeks.

Due to equation (2.11) and the initial conditions

$$\rho_k^{(n)}(0) = \delta_{kn} \tag{2.13}$$

the coefficients $\rho_k^{(n)}$ are different from zero if only the difference of the upper and lower indices $n-k$ is some multiple of the integer p. The set of equations (2.11) can be solved exactly in terms of the Gauss hypergeometric functions (which can be reduced to the complete elliptic integrals in some special cases) [18]. However, for our purposes here we do not need the explicit form of these solutions. It is sufficient to know the set of *generating functions* of an auxiliary variable z

$$R^{(k,j)}(z,\tau) = \sum_{l=-\infty}^{\infty} \rho_{j+pl}^{(j+pk)}(\tau)\,z^l \qquad j = 0, 1, \ldots, p-1 \quad k = 0, 1, \ldots. \tag{2.14}$$

Each of these functions satisfies the simple first-order partial differential equation

$$\frac{\partial R^{(n,j)}}{\partial \tau} = \left[\sigma\left(\frac{1}{z} - z\right) + 2\mathrm{i}\gamma\right]\left(j + pz\frac{\partial}{\partial z}\right) R^{(n,j)} \tag{2.15}$$

which is an immediate consequence of equations (2.11) and (2.14). The solution to equation (2.15) satisfying the initial condition $R^{(n,j)}(z,0) = z^n$ and the boundary condition $R^{(n,0)}(0,\tau) = 0$ reads [18]

$$R^{(k,j)}(z,\tau) = z^{-j/p}\left[\frac{zg(p\tau) + \sigma S(p\tau)}{g^*(p\tau) + z\sigma S(p\tau)}\right]^{k+j/p} - \delta_{j0}\left[\frac{\sigma S(p\tau)}{g^*(p\tau)}\right]^k \tag{2.16}$$

where

$$S(x) = \sinh(ax)/a \qquad g(x) = \cosh(ax) + \mathrm{i}\gamma S(x) \qquad a = \sqrt{1-\gamma^2}. \tag{2.17}$$

The consequences of equations (2.14) and (2.16) are the recurrence relations [18]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_m^{(n)} = n\left\{\sigma\left[\rho_m^{(n-p)} - \rho_m^{(n+p)}\right] + 2\mathrm{i}\gamma\rho_m^{(n)}\right\} \qquad n \geqslant p \quad \rho_m^{(0)} \equiv 0 \tag{2.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_m^{(n)} = n\left\{\sigma\left[\rho_{-m}^{(p-n)*} - \rho_m^{(p+n)}\right] + 2\mathrm{i}\gamma\rho_m^{(n)}\right\} \qquad n = 1, 2, \ldots, p-1 \tag{2.19}$$

which will also be used in the next sections. Formula (2.16) holds for any value of the detuning parameter γ. If $\gamma > 1$, then one should replace the functions $\sinh(ax)/a$ and $\cosh(ax)$ by their trigonometric counterparts $\sin(\tilde{a}x)/\tilde{a}$ and $\cos(\tilde{a}x)$, where $\tilde{a} = \sqrt{\gamma^2 - 1}$.

3. Energy density

Considering the systems with ideal moving boundaries one meets the problem of an ambiguity in the definition of the energy and the notion of the 'photons'. To avoid this difficulty we suppose that after some interval of time T the wall comes back to its initial position L_0 and no longer moves. Then for $t \geqslant T$ all ambiguities disappear, since the field operator assumes the form analogous to (2.4)

$$\hat{A}(x,t) = \sum_{m=1}^{\infty} \frac{2}{\sqrt{m}} \sin(\pi m x/L_0)\left[\hat{a}_m \mathrm{e}^{-\mathrm{i}\omega_m(t+\delta T)} + \text{h.c.}\right] \tag{3.1}$$

where new ('out') operators $\hat{a}_m$ are related to the initial operators $\hat{b}_n$ and $\hat{b}_n^\dagger$ by means of the Bogoliubov canonical transformation

$$\hat{a}_m = \sum_{n=1}^{\infty} \sqrt{\frac{m}{n}}\left[\hat{b}_n \rho_m^{(n)} - \hat{b}_n^\dagger \rho_{-m}^{(n)*}\right] \qquad m = 1, 2, \ldots \tag{3.2}$$

which preserves [18] the commutation relations $\left[\hat{a}_n, \hat{a}_k^\dagger\right] = \delta_{nk}$. The argument of the *constant* coefficients $\rho_{\pm m}^{(n)}(\tau)$ in (3.2) should be taken at the moment T, i.e. $\tau = \tau_T \equiv \frac{1}{2}\varepsilon\omega_1 T$. Due to our assumption, the parameter T can assume only discrete values: $T = N\pi/[p(1+\delta)]$ with an integer N. Consequently, the possible values of the argument τ are discrete, as well: $\tau = \tau^{(N)} = N\varepsilon\pi/[2p(1+\delta)]$. One should remember, however, that non-trivial physical effects can only be observed for the values $\tau \sim 1$ (or larger), i.e. for $N \sim \varepsilon^{-1} \gg 1$. In such a case the minimal increment $\Delta\tau = \tau^{(N+1)} - \tau^{(N)} \sim \varepsilon$ is so small that τ_T can be considered as a continuous variable (under the realistic conditions, $\varepsilon \leqslant 10^{-8}$ [6]). For this reason, we omit hereafter the subscript T, writing simply τ instead of τ_T or $\tau^{(N)}$.

The mean value of the energy density operator in one space dimension

$$\hat{W}(x,t) = \frac{1}{8\pi}\left[\left(\partial\hat{A}/\partial t\right)^2 + \left(\partial\hat{A}/\partial x\right)^2\right] \tag{3.3}$$

at $t \geqslant T$ equals (hereafter we assume $L_0 = 1$, i.e. $\omega_1 = \pi$)

$$\tilde{W}(x,t) = \pi \sum_{m,j=1}^{\infty} \sqrt{mj}\Big\{ \cos[\pi(m+j)x]\,\mathrm{Re}\left[\langle\hat{a}_m\hat{a}_j\rangle \mathrm{e}^{-\mathrm{i}\pi(m+j)t'}\right] + \tfrac{1}{2}\cos[\pi(m-j)x]\left[\langle\hat{a}_m^\dagger\hat{a}_j\rangle \mathrm{e}^{\mathrm{i}\pi(m-j)t'} + \langle\hat{a}_m\hat{a}_j^\dagger\rangle \mathrm{e}^{-\mathrm{i}\pi(m-j)t'}\right]\Big\} \tag{3.4}$$

where the quantum mechanical averaging $\langle\cdots\rangle$ is performed over the initial state of the field (the Heisenberg picture) and $t' \equiv t + \delta T$. The last term in (3.4) results in the divergent sum for $m = j$. Regularizing this divergence by means of the standard 'point-split' method [43] we obtain (see the appendix) the known expression for the one-dimensional negative vacuum Casimir energy [43–47]

$$\tilde{W}^{(Cas)} = -\pi/24 \qquad \left(\text{or } -\frac{\pi\hbar c}{24L_0^2} \text{ in the dimensional units}\right). \tag{3.5}$$

Extracting this vacuum energy from $\tilde{W}$ we arrive at the expression

$$W \equiv \tilde{W} - \tilde{W}^{(Cas)} = \pi \sum_{m,j=1}^{\infty} \sqrt{mj}\,\mathrm{Re}\left(\langle\hat{a}_m\hat{a}_j\rangle \cos[\pi(m+j)x]\,\mathrm{e}^{-\mathrm{i}\pi(m+j)t'} + \langle\hat{a}_m^\dagger\hat{a}_j\rangle \cos[\pi(m-j)x]\,\mathrm{e}^{\mathrm{i}\pi(m-j)t'}\right). \tag{3.6}$$

The same expression (3.6) can be obtained if one calculates the mean value of the *normally ordered* (with respect to the operators $\hat{a}_n^\dagger$ and $\hat{a}_n$) counterpart of the operator (3.3) (cf [28]). Then the total energy (without the vacuum part) assumes the usual form

$$\mathcal{E} = \int_0^{L_0} W(x,t)\,\mathrm{d}x = \sum_{n=1}^{\infty} \omega_n \left\langle\hat{a}_n^\dagger\hat{a}_n\right\rangle \tag{3.7}$$

which justifies the choice of the normalization in (2.1) and (3.1).

Since the initial quantum state was defined with respect to the 'in' operators $\hat{b}_n^\dagger$ and $\hat{b}_n$, we must express the 'out' operators $\hat{a}_m^\dagger$ and $\hat{a}_m$ in terms of $\hat{b}_n^\dagger$ and $\hat{b}_n$ by means of formula (3.2). Thus we arrive at the expression containing a combination of the mean values $\langle\hat{b}_n\hat{b}_k\rangle$, $\langle\hat{b}_n^\dagger\hat{b}_k^\dagger\rangle$, $\langle\hat{b}_n^\dagger\hat{b}_k\rangle$ and $\langle\hat{b}_n\hat{b}_k^\dagger\rangle$ calculated in the initial quantum state. For the initial vacuum state defined according to the relations $\hat{b}_n|0\rangle = 0$, $n = 1, 2, \ldots$, the only non-zero mean values are $\langle\hat{b}_n\hat{b}_k^\dagger\rangle = \delta_{nk}$. Then (3.6) is transformed into the triple sum

$$W_0(x,t) = \pi \sum_{n,m,j=1}^{\infty} \frac{mj}{n}\,\mathrm{Re}\left\{\cos[\pi(m-j)x]\,\mathrm{e}^{\mathrm{i}\pi(m-j)t'}\rho_{-m}^{(n)}\rho_{-j}^{(n)*} - \cos[\pi(m+j)x]\,\mathrm{e}^{\mathrm{i}\pi(m+j)t'}\rho_{-m}^{(n)}\rho_{j}^{(n)*}\right\}. \tag{3.8}$$

Evidently, $W_0(x,t) = 0$ for $t \leqslant 0$. For an arbitrary initial state the energy density can be written as a sum of the 'vacuum' and 'non-vacuum' contributions

$$W = W_0 + W_1 \qquad W_1 = \pi \sum_{n,k=1}^{\infty} \frac{1}{\sqrt{nk}}\,\mathrm{Re}\left[\langle\hat{b}_n\hat{b}_k\rangle B^{(nk)} + \langle\hat{b}_n^\dagger\hat{b}_k\rangle \tilde{B}^{(nk)}\right] \tag{3.9}$$

where

$$B^{(nk)} = \sum_{m,j=1}^{\infty} mj\Big\{\cos[\pi(m+j)x]\big[\mathrm{e}^{-\mathrm{i}\pi(m+j)t'}\rho_m^{(n)}\rho_j^{(k)} + \mathrm{e}^{\mathrm{i}\pi(m+j)t'}\rho_{-m}^{(n)}\rho_{-j}^{(k)}\big] - \cos[\pi(m-j)x]\big[\mathrm{e}^{-\mathrm{i}\pi(m-j)t'}\rho_m^{(n)}\rho_{-j}^{(k)} + \mathrm{e}^{\mathrm{i}\pi(m-j)t'}\rho_{-m}^{(n)}\rho_j^{(k)}\big]\Big\} \tag{3.10}$$

$$\tilde{B}^{(nk)} = \sum_{m,j=1}^{\infty} mj\Big\{\cos[\pi(m-j)x]\big[\mathrm{e}^{-\mathrm{i}\pi(m-j)t'}\rho_{-m}^{(n)*}\rho_{-j}^{(k)} + \mathrm{e}^{\mathrm{i}\pi(m-j)t'}\rho_m^{(n)*}\rho_j^{(k)}\big] - \cos[\pi(m+j)x]\big[\mathrm{e}^{-\mathrm{i}\pi(m+j)t'}\rho_{-m}^{(n)*}\rho_j^{(k)} + \mathrm{e}^{\mathrm{i}\pi(m+j)t'}\rho_m^{(n)*}\rho_{-j}^{(k)}\big]\Big\}. \tag{3.11}$$

Making the change of the summation index $j \to -j$ in the first term of (3.8) we can write

$$W_0(x,t) = -\pi\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{j=-\infty}^{\infty}\frac{mj}{n}\cos[\pi(m+j)x]\,\mathrm{e}^{\mathrm{i}\pi(m+j)t'}\rho_{-m}^{(n)}\rho_j^{(n)*}.$$

Similarly, changing the indices $m \to -m$ or $j \to -j$ in (3.10) and (3.11) we can reduce four sums with apparently different summands and the indices running from 1 to ∞ to the unified sums whose two indices run from $-\infty$ to ∞:

$$B^{(nk)} = \sum_{m,j=-\infty}^{\infty} mj\cos[\pi(m+j)x]\,\mathrm{e}^{-\mathrm{i}\pi(m+j)t'}\rho_m^{(n)}\rho_j^{(k)}$$

$$\tilde{B}^{(nk)} = \sum_{m,j=-\infty}^{\infty} mj\cos[\pi(m-j)x]\,\mathrm{e}^{\mathrm{i}\pi(m-j)t'}\rho_m^{(n)*}\rho_j^{(k)}.$$

Now, replacing the cosine function by the sum of two imaginary exponentials we see that $W(x,t)$ is actually the sum of two identical functions of the lightcone variables:

$$W(x,t) = \tfrac{1}{2}\pi\,[F(u;\tau) + F(v;\tau)] \qquad u = t' + x \quad v = t' - x \tag{3.12}$$

where

$$F = F_0 + \sum_{n,k=1}^{\infty}\frac{1}{\sqrt{nk}}\,\mathrm{Re}\left[\langle\hat{b}_n\hat{b}_k\rangle F^{(nk)} + \langle\hat{b}_n^{\dagger}\hat{b}_k\rangle\tilde{F}^{(nk)}\right] \tag{3.13}$$

$$F_0(u;\tau) = -\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{j=-\infty}^{\infty}\frac{mj}{n}\mathrm{e}^{\mathrm{i}\pi(m+j)u}\rho_{-m}^{(n)}(\tau)\rho_j^{(n)*}(\tau) \tag{3.14}$$

$$F^{(nk)}(u;\tau) = \sum_{m=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} mj\mathrm{e}^{-\mathrm{i}\pi(m+j)u}\rho_m^{(n)}(\tau)\rho_j^{(k)}(\tau) \tag{3.15}$$

$$\tilde{F}^{(nk)}(u;\tau) = \sum_{m=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} mj\mathrm{e}^{\mathrm{i}\pi(m-j)u}\rho_m^{(n)*}(\tau)\rho_j^{(k)}(\tau). \tag{3.16}$$

The extra argument τ in the above expressions is introduced in order to emphasize that the energy density depends not only on the value of the current time variable t (which must satisfy the condition $t > T$), but also on the moment of time T when the wall stopped moving. It is worth mentioning that the variables t and τ are independent, as well as u and τ or v and τ.

Evidently, the double sums (3.15) and (3.16) are factorized to the products of independent sums over m and j:

$$F^{(nk)}(u;\tau) = G^{(n)}(u;\tau)G^{(k)}(u;\tau) \qquad \tilde{F}^{(nk)}(u;\tau) = G^{(n)*}(u;\tau)G^{(k)}(u;\tau)$$
$$G^{(n)}(u;\tau) = \sum_{m=-\infty}^{\infty} m\mathrm{e}^{-\mathrm{i}\pi mu}\rho_m^{(n)}(\tau) = \frac{\mathrm{i}}{\pi}\frac{\partial}{\partial u}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\pi mu}\rho_m^{(n)}(\tau). \tag{3.17}$$

The last sum in (3.17) can be easily expressed in terms of the generating function (2.14) if one writes $n = j + kp$, $m = j + lp$ and $z = \exp(-\mathrm{i}\pi pu)$. Thus we obtain

$$G^{(n)}(u;\tau) = \frac{nz\,[zg(p\tau) + \sigma S(p\tau)]^{n/p-1}}{[g^*(p\tau) + z\sigma S(p\tau)]^{n/p+1}}\bigg|_{z=\exp(-\mathrm{i}\pi pu)} = nf^{1/2}\Lambda^{n/p} \tag{3.18}$$

where

$$f(u;\kappa) = \left|g^*(p\tau) + z\sigma S(p\tau)\right|^{-4} = \frac{\left(1-\kappa^2\right)^2}{\left[1+\kappa^2+2\sigma\kappa\cos(p\pi u-\varphi)\right]^2} \tag{3.19}$$

$$\Lambda = \frac{zg(p\tau)+\sigma S(p\tau)}{g^*(p\tau)+z\sigma S(p\tau)} = \mathrm{e}^{\mathrm{i}(2\varphi-\pi pu)}\frac{1+\sigma\kappa\exp[\mathrm{i}(\pi pu-\varphi)]}{1+\sigma\kappa\exp[\mathrm{i}(\varphi-\pi pu)]} \tag{3.20}$$

$$\kappa = \frac{S(p\tau)}{\sqrt{1+S^2(p\tau)}} \qquad \exp(\mathrm{i}\varphi) = \sqrt{1-\gamma^2\kappa^2}+\mathrm{i}\gamma\kappa. \tag{3.21}$$

In the 'vacuum' contribution (3.14) we have some asymmetry between the indices m and j, since m runs from 1 to ∞, whereas j runs from $-\infty$ to ∞. This asymmetry can be eliminated if one differentiates both sides of equation (3.14) with respect to the independent variable τ at a fixed value of u and performs the summation over the superscript n with the aid of the recurrence relations (2.18) and (2.19). It is easy to verify that all the summands with $n \geqslant p$ are cancelled, so the infinite series over n can be reduced to the finite sum from 1 to $(p-1)$:

$$\frac{\partial F_0(u;\tau)}{\partial\tau} = -\sigma\,\mathrm{Re}\sum_{n=1}^{p-1}\sum_{m=1}^{\infty}\sum_{j=-\infty}^{\infty} mj\mathrm{e}^{\mathrm{i}\pi(m+j)u}\left[\rho_{-m}^{(n)}(\tau)\rho_{-j}^{(p-n)}(\tau)+\rho_m^{(p-n)*}(\tau)\rho_j^{(n)*}(\tau)\right]. \tag{3.22}$$

Making the change of summation indices $m \to -m$, $j \to -j$, $n \to p-n$ in the first product inside the square brackets one can reduce two sums in the right-hand side of (3.22) to the single series where both the indices m and j run from $-\infty$ to ∞. Moreover, the sums over m and j become completely independent, giving rise to the equation

$$\frac{\partial F_0(u;\tau)}{\partial\tau} = -\sigma\,\mathrm{Re}\sum_{n=1}^{p-1} G^{(n)}(u;\tau)G^{(p-n)}(u;\tau) \tag{3.23}$$

where $G^{(n)}(u;\tau)$ is given by (3.17). Due to (3.18) the sum in the right-hand side of (3.23) is reduced to the sum $\sum_{n=1}^{p-1} n(p-n) = \frac{1}{6}p(p^2-1)$. Introducing the variable $\eta = \exp(2ap\tau)$ we obtain the explicit expression

$$\frac{\partial F_0(u;\eta)}{\partial\eta} = -\frac{(p^2-1)a^4\eta\left[\eta^2(1+\alpha+\beta)+\alpha-\beta-1\right]}{12\left[\eta^2(1+\alpha+\beta)-2\eta(\gamma^2+\beta)+1+\beta-\alpha\right]^3} \tag{3.24}$$

where $\alpha = \sigma a\cos(p\pi u)$ and $\beta = \sigma\gamma\sin(p\pi u)$. Integrating (3.24) with the initial condition $F_0 = 0$ at $\tau = 0$ (or $\eta = 1$) we arrive after some algebra at the simple expression

$$F_0(u;\kappa) = \mathcal{B}\,[f(u;\kappa)-1] \qquad \mathcal{B} \equiv \left(p^2-1\right)/24 \tag{3.25}$$

where the function $f(u;\kappa)$ is given by (3.19). Finally, we obtain the following expression for the function $F(u;\tau)$ defined by equation (3.12):

$$F = -\mathcal{B} + f(u;\tau)\left\{\mathcal{B} + \sum_{n,k=1}^{\infty}\sqrt{nk}\,\mathrm{Re}\left[\langle\hat{b}_n\hat{b}_k\rangle\Lambda^{(n+k)/p} + \langle\hat{b}_n^\dagger\hat{b}_k\rangle\Lambda^{(k-n)/p}\right]\right\}. \tag{3.26}$$

In the special case of the initial states whose density matrix is diagonal in the Fock basis, so that $\langle\hat{b}_n^\dagger\hat{b}_k\rangle = \nu_n\delta_{nk}$ and $\langle\hat{b}_n\hat{b}_k\rangle = 0$ (for example, the Fock or thermal states; ν_n is the mean number of quanta in the nth mode), the sum in (3.26) is proportional to the initial total energy $\mathcal{E}_0$ in all the modes (above the Casimir level):

$$F^{(diag)}(u;\tau) = -\mathcal{B} + f(u;\tau)\,[\mathcal{B} + \mathcal{N}_0] \qquad \mathcal{N}_0 = \sum_{n=1}^{\infty} n\nu_n = \mathcal{E}_0/\pi. \tag{3.27}$$

The total energy at $\tau \geqslant 0$ is obtained by integrating the density $W(x)$ (3.12) over x. The contribution of the vacuum (function F_0 in (3.13)) and 'diagonal' terms (given by the partial sum in (3.13) over $n = k$) can be calculated with the aid of the formula

$$\int_0^\pi \frac{\mathrm{d}x}{(a + b\cos x)^2} = \frac{\pi a}{\left(a^2 - b^2\right)^{1/2}}$$

which is a simple consequence of the integrals given in [48]. To find the contribution of 'non-diagonal' terms ($n \neq k$) it is convenient to replace the integration over x by the integration in the complex z-plane ($z = \exp[-\mathrm{i}\pi pu]$ or $z = \exp[-\mathrm{i}\pi pv]$) over the circle $|z| = 1$. One can check that this circle is passed p times when x goes from 0 to 1 (if one takes into account both the 'u'- and 'v'-contributions). It turns out that the integrals of the 'non-diagonal' terms are different from zero provided the corresponding integrands in the z-plane have simple poles inside the circle $|z| = 1$. This happens only when $k + n = p$ in the first term inside the square brackets in (3.26) and $k - n = p$ in the second term inside the same brackets. This fact becomes clear if one looks in figure 1, which illustrates the principal difference in the behaviour of the function

$$K_p^\mu(x) = \mathrm{Re}\left(f(u)[\Lambda(u)]^{\mu/p} + f(v)[\Lambda(v)]^{\mu/p}\right) \tag{3.28}$$

for $\mu = p$ and $\mu \neq p$. It is obvious that the integral of the function $K_p^\mu(x)$ over the interval $0 < x < 1$ equals zero if $\mu \neq p$ (we have chosen the values $\mu = 5$ and $p = 2$ to give an example).

Finally, we obtain exactly the same expression which was found in a different way in [18]:

$$\mathcal{E}(\tau) = \mathcal{E}_0 + 2S^2(p\tau)\left[\mathcal{E}_0 + \pi\mathcal{B} + \tfrac{1}{2}\gamma\sigma\,\mathrm{Im}(\mathcal{G})\right] - \tfrac{1}{2}\sigma S(2p\tau)\,\mathrm{Re}(\mathcal{G}) \tag{3.29}$$

where

$$\mathcal{G} = 2\pi\sum_{n=1}^{\infty}\sqrt{n(n+p)}\langle\hat{b}_n^\dagger\hat{b}_{n+p}\rangle + \pi\sum_{n=1}^{p-1}\sqrt{n(p-n)}\langle\hat{b}_n\hat{b}_{p-n}\rangle. \tag{3.30}$$

4. Packet formation

Now let us analyse the expressions for the energy density obtained in section 3. For the initial vacuum state we see immediately from equations (3.19) and (3.25) that in the generic case the function $W_0(x,t)$ with the fixed value of the 'fast time' t has p peaks in the interval $0 \leqslant x \leqslant 1$, whose positions are determined by the equations $\sigma\cos(p\pi u - \varphi) = -1$ and $\sigma\cos(p\pi v - \varphi) = -1$ (see figure 2). Obviously, for $t > T$ the energy density is a periodic

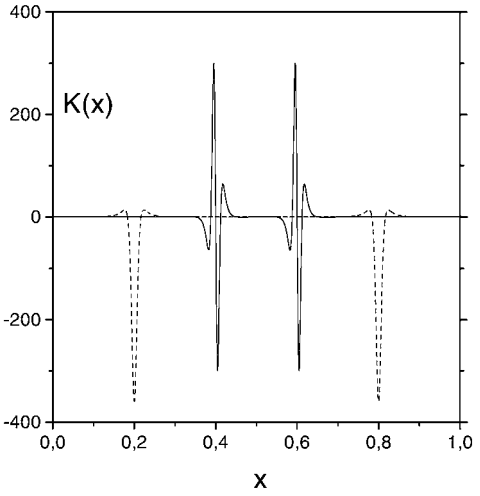

Figure 1. The functions $K_2^2(x)$ (broken curve) and $K_2^5(x)$ (full curve) defined by equation (3.28) for $\varphi = 0$ and $\kappa = 0.9$. The fractional parts of the 'fast time' t are chosen as follows: $[t] = 0.3$ for K_2^2 and $[t] = 0.1$ for K_2^5 (since the integral part of t is not important, we omit it in all the figures).

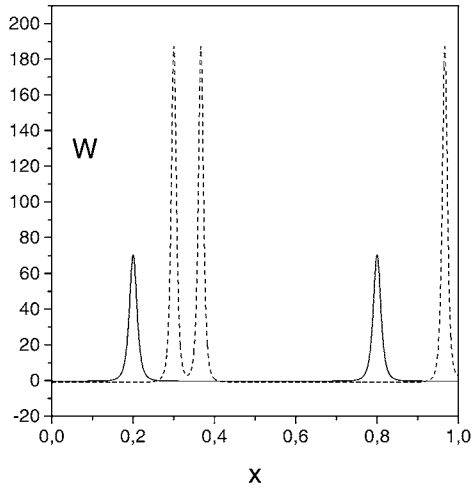

Figure 2. The energy density for the initial vacuum state in the case of the strict resonance $\gamma = 0$, for $[t] = 0.3$ and $\kappa = 0.9$; $p = 2$ (full curve) and $p = 3$ (broken curve).

function of the time variable t, with the period $\Delta t = 1$ if p is an even number and $\Delta t = 2$ if p is odd. For this reason, in all the figures we give only the fractional part $[t]$ of t and omit the inessential integral part (which is very large in fact: for the maximal possible value $\varepsilon \sim 10^{-8}$ [6] we have $t \sim 10^8$ if $\tau \sim 1$).

For the even values of the resonance multiplicity p we have $p/2$ peaks moving (with the speed of light) in the positive direction and $p/2$ peaks moving in the negative direction. If p is odd, then the numbers of peaks of each kind differ by 1. All the peaks have the same height

$$W_{max}^{(vac)} = 2\pi\mathcal{B}\kappa/(\kappa-1)^2 = \tfrac{1}{2}\pi\mathcal{B}\left(\mathrm{e}^{4p\tau}-1\right) \tag{4.1}$$

(in this section the expressions containing τ are related to the special case of the strict resonance $\gamma = 0$), except for some distinguished instants of time when two peaks moving in the opposite directions merge, forming a peak with double the height.

If $\kappa \to 1$ (i.e. $\tau > 1$ and $\gamma < 1$), then the energy density can be approximated in the vicinity of each peak by the Lorentz-like distribution

$$W^{(vac)}(\delta x) = \frac{W_{max}^{(vac)}}{\left[1+\left(2\delta x/\Delta_{1/4}\right)^2\right]^2} \qquad \Delta_{1/4} = \frac{2}{p\pi}\frac{1-\kappa}{\sqrt{\kappa}} \approx \frac{4}{p\pi}\mathrm{e}^{-2p\tau} \tag{4.2}$$

where the width $\Delta_{1/4}$ of each peak is defined as the double distance between the position of the maximum and the point where the energy density decreases four times. One can also introduce the 'energy width' of each peak by means of the relation $W_{max}\Delta_E = \mathcal{E}(\tau)/p$. For $\kappa \to 1$ we obtain $\Delta_E \approx (1-\kappa)/(2\pi p) \approx (\pi p)^{-1}\mathrm{e}^{-2p\tau}$.

Except for narrow regions of length $\Delta_+ \approx (\pi p)^{-1}\sqrt{1-\kappa} \approx \sqrt{2}(\pi p)^{-1}\mathrm{e}^{-p\tau}$ nearby the peaks the 'dynamical' energy density is less than its initial vacuum value, in agreement with the results of [28, 30, 36, 37] obtained within the framework of different approaches. The

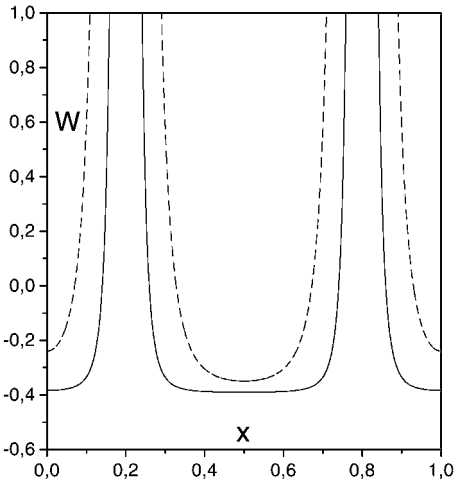

Figure 3. The energy density for the initial vacuum (full curve) and thermal (broken curve) states, for $\gamma = 0$, $p = 2$, $[t] = 0.3$ and $\kappa = 0.9$. The initial total energy of the thermal state is $\mathcal{E}_0 = \pi$.

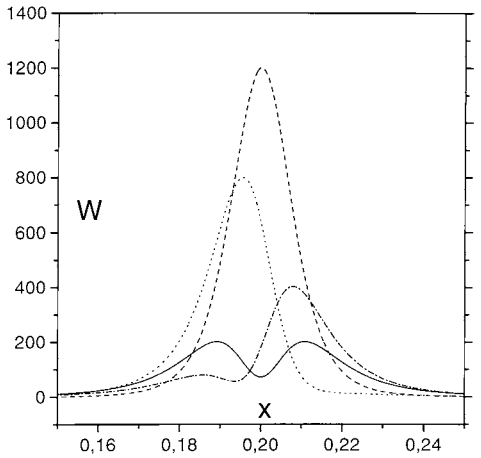

Figure 4. The 'fine structure' of the energy density peak in the case when initially the first mode was in the coherent state with $|\alpha| = 1$ but different values of the phase ψ, for $\gamma = 0$, $p = 2$, $[t] = 0.3$ and $\kappa = 0.9$. The curves are ordered according to the increase of the maximal heights as follows: $\psi = 0$ (full curve), $\psi = 9\pi/10$ (chain curve), $\psi = \pi/4$ (dotted curve), $\psi = \pi/2$ (broken curve).

minimum values of W_0 far off the peaks are given by (taking into account the contributions of both the functions $F_0(u)$ and $F_0(v)$)

$$W_{min} = -4\pi\mathcal{B}\frac{\kappa}{(\kappa+1)^2} = \pi\mathcal{B}\left(\mathrm{e}^{-4p\tau} - 1\right). \tag{4.3}$$

If $\kappa \to 1$, $W_{min} \to -\pi\left(p^2 - 1\right)/24$. Adding to this expression the initial Casimir energy (3.5) we obtain the total asymptotical minimum value (cf [28])

$$\tilde{W}_{min}^{(as)} = -\pi p^2/24. \tag{4.4}$$

For an arbitrary initial state the energy density has, besides the 'vacuum' part, the additional terms given in equation (3.26). Since these terms are proportional to the same functions $f(u;\kappa)$ or $f(v;\kappa)$ which determine the structure of the 'vacuum' part, the positions of the peaks are not changed (remember that $|\Lambda| = 1$). For the initial states with diagonal density matrices in the Fock basis (in particular, for the thermal states) all the peaks still have equal heights, increased by the quantity $\Delta W_{max} = \frac{1}{2}\mathcal{E}_0(1+\kappa)^2/(1-\kappa)^2$, compared with the vacuum case. However, the asymptotic *minimal* value of the energy density at $\kappa \to 1$ does not depend on the initial state, as it is given by formula (4.4) in all cases. Figure 3 shows in detail the behaviour of the energy density in the regions where it is negative, for the vacuum and thermal initial states.

If the initial density matrix in the Fock basis has non-zero off-diagonal elements (as happens, in particular, for any pure state different from the Fock one, e.g. for the coherent states), different terms in the sum (3.26) can interfere. Consequently, the peaks acquire some kind of 'fine structure'. For example, if only the first mode was excited initially in the coherent

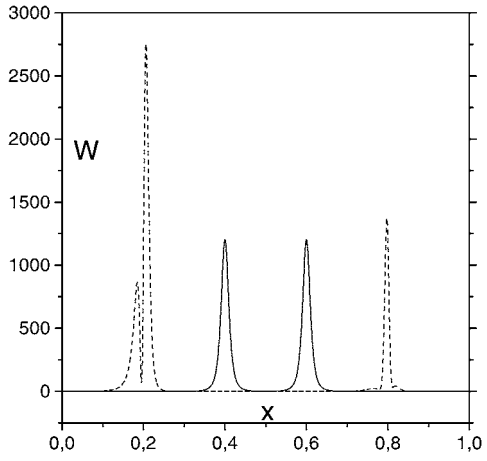

Figure 5. The energy density profiles for two different initial states with the same value of the total energy $\mathcal{E}_0 = 2\pi$. The broken curve with two differently deformed peaks corresponds to the case when two first modes were in the coherent states with the same amplitude $|\alpha_1| = |\alpha_2| = 1$, but different phases $\psi_1 = 0$, $\psi_2 = \pi/4$. The full curve with two identical peaks corresponds to the initial thermal state. In both cases $\gamma = 0$, $p = 2$ and $\kappa = 0.9$. In order to separate the peaks we choose $[t] = 0.3$ for the coherent states, but $[t] = 0.1$ for the thermal state.

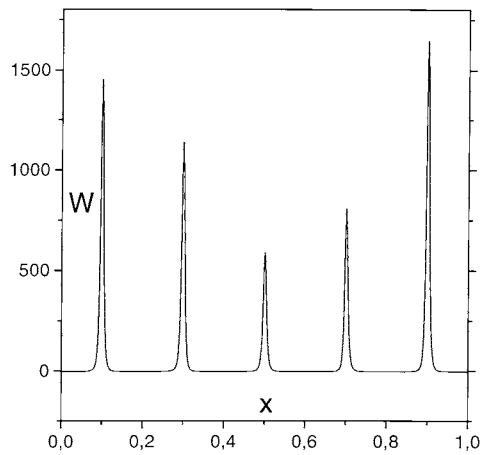

Figure 6. The energy density in the case when initially the first mode was in the coherent state with $|\alpha| = 1$ and $\psi = \pi/4$, for $\gamma = 0$, $p = 5$, $[t] = 0.3$ and $\kappa = 0.9$.

state $|\alpha\rangle$, $\alpha = |\alpha| \exp(\mathrm{i}\psi)$, then for $p = 2$ and $\gamma = 0$ (the strict resonance) we have

$$\Delta W \equiv W - W^{(vac)} = \pi |\alpha|^2 \frac{\left(1-\kappa^2\right)^2 \left[\kappa \sin(z+\psi) + \sin(z-\psi)\right]^2}{\left[(1-\kappa)^2 + 4\kappa \sin^2 z\right]^3}$$

where $z \equiv \pi\,(u - u_*)$ and u_* is the position of the 'vacuum' peak determined above. If $\psi = \pi/2$, then we have the high maximum $\Delta W^{max}_{\pi/2} = \pi|\alpha|^2(1+\kappa)^2/(1-\kappa)^2$ at $z = 0$. However, if $\psi = 0$, then instead of a maximum we have the minimum $\Delta W = 0$ at the same point $z = 0$, and the peak is split in two symmetric humps with equal maximal heights $\Delta W^{max}_0 = \left([1+\kappa]^2/27\kappa\right) \Delta W^{max}_{\pi/2}$ located at the points $\sin z = \pm(1-\kappa)/\sqrt{8\kappa}$. In the intermediate case $0 < \psi < \pi/2$ asymmetric forms of the peaks are observed (see figure 4).

If $p > 2$ or several modes were excited initially, the interference between different terms in (3.26) can result in different heights of the peaks and more complicated 'fine structures' (provided $\langle \hat{b}_n \hat{b}_k \rangle \neq 0$ for some n and k) (see figures 5 and 6).

If the detuning γ is different from zero, then some deformations of the form of peaks are observed, although the maximal heights are still of the same order of magnitude as in the case $\gamma = 0$, as far as $\gamma \leqslant 1$ (see figure 7). However, if the detuning exceeds the critical value $\gamma = 1$, the energy becomes an oscillating function of the 'slow time' τ, the amplitude of oscillations being proportional approximately to $\left(\gamma^2 - 1\right)^{-1}$ [18, 24]. This situation is shown in figure 8. The peaks become rather wide and low, since the parameter κ is limited by the inequality $\kappa \leqslant \gamma^{-1}$ if $\gamma > 1$.

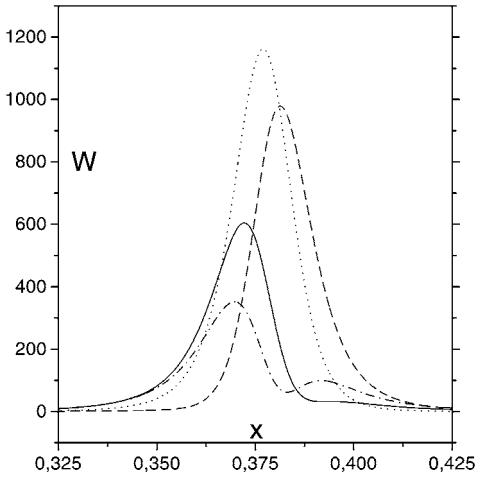

Figure 7. The 'fine structure' of the energy density peak in the 'critical resonance' case $\gamma = 1$, when initially the first mode was in the coherent state with $|\alpha| = 1$ but different values of the phase ψ, for $p = 2$, $[t'] = 0.3$ and $\kappa = 0.9$. The curves are ordered according to the increase of the maximal heights as follows: $\psi = 9\pi/10$ (chain curve), $\psi = 0$ (full curve), $\psi = \pi/2$ (broken curve), $\psi = \pi/4$ (dotted curve).

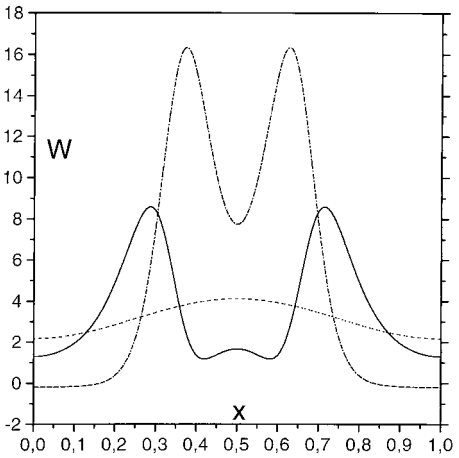

Figure 8. The energy density in the 'off-resonance' case $\gamma = 2$, when initially the first mode was in the coherent state with $|\alpha| = 1$ but different values of the phase $\psi = 0$ (full curve) and $\psi = \pi/2$ (chain curve with the highest maximal value), for $p = 2$, $[t'] = 0.3$ and $\kappa = 0.4$. The broken curve with a smooth maximum in the centre gives the initial energy density distribution (corresponding to the value $\kappa = 0$).

5. Discussion

Let us summarize the main results of the paper. We have found simple explicit analytical expressions describing the energy density distribution of the field inside an ideal 1D cavity whose walls performed harmonic vibrations at some (quasi)resonance frequency, after the walls finally came to rest. These expressions show that the energy density is concentrated in the form of p sharp peaks (either in space for a fixed time moment, or in time at the fixed point inside the cavity), provided the resonance condition $\gamma \leqslant 1$ is fulfilled, where γ is the ratio of the dimensionless detuning δ to the dimensionless amplitude of the wall oscillations ε and p is the multiplicity of the resonance with respect to the fundamental field eigenfrequency. These peaks move with the speed of light from one wall to another. The shapes of the peaks are sensitive to the initial state of the field: rather simple symmetric Lorentzian-like profiles are observed for the vacuum or thermal states, whereas for other states we have asymmetric peaks possessing some kind of 'fine structure'.

Since in the single space dimension the components of the energy–momentum tensor T_{00} and T_{11} are given by similar expressions, the force acting on each wall has the same time dependence as the energy density at the points $x = 0$ and 1. For most of the time during the period of field oscillations $2L_0/c$ (where L_0 is the distance between the walls at rest) this force is negative, being less than the static Casimir force, with the maximal amplification coefficient p^2. However, the average value of the force over the period is positive due to the creation of real photons inside the cavity.

If the walls possess some small transmission coefficient, then a small part of the radiation accumulated inside the cavity can leave it. In this case one could observe sharp pulses of radiation outside the cavity [16], whose amplitudes must be proportional to the heights of the peaks inside the cavity multiplied by the small transmission coefficient. The intensity of these

pulses can be significantly increased, if the initial state is different from vacuum and possesses sufficient energy, like thermal states [16, 26] or coherent states. However, to describe the form of the pulses exactly it is necessary to develop a more general theory which would take into account the boundary conditions corresponding to the partially transmitting walls (because the non-zero transmission coefficient can change the pulse shape significantly, just as the non-zero detuning deformed the form of packets in the examples considered in section 4).

The total energy $\mathcal{E}$ of the field depends on the parameter $\tau = \frac{1}{2}\varepsilon\omega_1 T = \pi\varepsilon\mathcal{N}$, where $\mathcal{N}$ is the number of full oscillations performed by the wall. Due to equation (3.29), the dynamical contribution to the energy exceeds the absolute value of the static Casimir energy if $\tau > \tau_c \approx \left[2p^2\left(p^2-1\right)\right]^{-1/2} \approx \left(p^2\sqrt{2}\right)^{-1}$, and it grows exponentially if $p\tau > 1$. However, the level $\mathcal{E} > \hbar\omega_1$ can be achieved only after a great number of oscillations, since the crucial parameter ε is very small even in the most optimistic situations. It was shown in [6, 32] that the critical value of this parameter equals $\varepsilon_{max} \sim (v_s/2\pi c)\xi_{max} \sim 10^{-8}$, otherwise the vibrating wall will be destroyed because of the immense internal mechanical stresses (here v_s is the sound velocity inside the wall and $\xi_{max} \sim 10^{-2}$ is the maximal possible non-destructive deformation). The maximal possible velocity of the surface is [6] $v_{max} \sim \xi_{max} v_s \sim 50$ m s^{-1}. Note that v_{max} and ε_{max} do not depend on the fundamental eigenfrequency ω_1.

Evidently, the increase of the energy cannot be unlimited even when $\mathcal{N} \to \infty$, due to at least two circumstances. The first one is that our solutions hold provided $\mathcal{N} < \varepsilon^{-2}$. For larger values of $\mathcal{N}$ the nonlinear effects will destroy the effective intermode interaction. These effects, however, are quite unimportant in realistic situations, due to the smallness of the parameter ε (remember the discussion in section 2). Much more important are the limitations due to inevitable losses in the cavity walls. To overcome these losses one needs a cavity with the quality factor $Q > \varepsilon_{max}^{-1} > 10^8$. The unsolved problem is how to excite the vibrations of the wall surface at high frequencies of the order of 1–10 GHz (corresponding to a cavity length of the order of centimetres), not speaking of the optical frequencies, required for microcavities with dimension $L_0 \sim 1$ μm.

One of the reasons for the studies on the dynamical Casimir effect over the last few years has been Schwinger's hypothesis [9, 49], which could explain the *sonoluminescence* phenomenon, i.e. the emission of bright short pulses of visible light from gas bubbles in water, when the bubbles pulsate due to the pressure oscillations in a strong standing acoustic wave (see, e.g., [50, 51] for a review and references). Although our results, obtained in the framework of a simplified one-dimensional model, cannot be applied directly to the analysis of this problem, they are not in favour of Schwinger's hypothesis. The main difficulty is connected with the quite different time scales of the phenomena. The accumulation of the 'dynamic Casimir energy' is a very slow process, which needs a great number of wall oscillations, whereas the sonoluminescence pulses (containing up to 10^7 photons) have a duration of the order of picoseconds. Moreover, the wall oscillations must be in extremely fine-tuned resonance with the field eigenfrequencies, since the detuning $\delta > \varepsilon$ completely destroys the energy growth [18]. In particular, if the frequency of the wall oscillations ω_{wall} is much less than the minimal field eigenfrequency ω_1, then the field variation is adiabatic, and the mean number of photons created is proportional to [6] $\varepsilon^2\,(\omega_{wall}/\omega_1)^4 \ll 1$. These features survive in the three-dimensional model, too [17]. It is difficult to believe that the very specific conditions described above could arise naturally in the sonoluminescence case (see also the discussion in [52]).

Actually, the main obstacle to producing the 'Casimir light' is the very low ratio of the wall velocity to the speed of light in possible laboratory experiments. If the velocity of the boundary were of the order of c, then a sufficient number of photons could be created from the vacuum practically for any law of motion. For the non-relativistic velocities the only possibility is

to accumulate the effect gradually under the resonance conditions. Nonetheless, one cannot exclude that the experimental situation could be improved in the case of using some kinds of 'effective mirrors', e.g. layers made from an electron–hole plasma, as was suggested in [53].

Acknowledgment

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Appendix. Renormalization of the vacuum energy

Due to the commutation relations $\left[\hat{a}_m, \hat{a}_j^\dagger\right] = \delta_{mj}$ the series (3.4) contains the vacuum divergent 'diagonal' ($m = j$) part

$$\tilde{W}^{(vac)} = (\pi/2)\sum_{m=1}^{\infty} m \exp(-\mathrm{i}\pi m t' + \mathrm{i}\pi m t'). \tag{A.1}$$

The recipe of how to regularize this divergence was given in [43]. One should write the first term in the argument of the exponential in (A.1) as it stands, but replace t' in the second term by $t' + \mathrm{i}\eta$, $\eta > 0$ (the 'point-splitting method'). Then the sum becomes convergent, giving

$$\tilde{W}^{(vac)}(\eta) = (\pi/2)\sum_{m=1}^{\infty} m\mathrm{e}^{-m\pi\eta} = (\pi/8)\left[\sinh(\pi\eta/2)\right]^{-2}.$$

The Taylor expansion of this function reads $\tilde{W}^{(vac)}(\eta) = (2\pi\eta^2)^{-1} - \pi/24 + \mathcal{O}(\eta^2)$. According to [43], one should remove the divergent term $(2\pi\eta^2)^{-1}$ and after that proceed to the limit $\eta \to 0$. This limit value is exactly the Casimir energy density (3.5), which does not depend on the coordinate x in the case involved.

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